On Fast High-Precision Evaluation of Some Multi-Dimensional Integrals with Polynomial Complexity: Using CAS

Extended Abstract

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There is a wide variety of techniques for numeric computation of a multi-dimensional integral (MDI). The choice of the technique to apply depends rather on the integration region than on dimension value of space to integrate. To put it approximately, if one confines oneself by 'low'-dimensional spaces, near, say, up to 10-dimensional ones, and a modest accuracy, say, about 10 decimal digits, then it is possible to apply successfully 1) the (Quasi) Monte Carlo methods family (QMC) which are based on usage of (non)-pseudorandom numbers – especially for 'complicated', 'irregular' integration regions; 2) the multi-dimensional quadrature methods family constructed from several one- or low-dimensional quadratures by a 'direct product' operation (DP) (especially for 'simple' , 'regular' regions such as hyper-parallelepiped, generalized ellipsoid or its surface, direct product of a k-dimensional sphere by l-dimensional cube etc).

Any good numeric library like NAG Fortran Library includes a few standard subroutines in which algorithms are implemented of both MC and DP classes for computing of above-mentioned integrals, e.g., D01FxF within the NAG package. Note that one can usually get, at most, double precision, some 16 decimal digits, with such a kind of software.

In 'high'-dimensional spaces, when a large accuracy is required, the mentioned techniques will fail because of their exponential computational complexity, for example, if one needs to evaluate an MDI in 1000-dimensional space with accuracy of 1000 decimal digits. Certainly, it is impossible to do for an arbitrary integration region and an arbitrary integrand. However, there are a number of MDIs taken over 'simple' regions (hyper-rectangle, n-sphere, n-simplex ...) which can be either reduced to 1- or 2-dimensional integrals or even evaluated exactly in elementary and/or special functions. In the latter case, one can use a computer algebra system (CAS) to evaluate the MDI under solution symbolically and then to approximate the output to the required accuracy.

Unfortunately, the total number of the known MDIs expessible through a closed form is still pretty little at present. So it is natural to try to find some new formulas of this kind.

If dimension of space is high enough (near 100 or greater) then it becomes, in view of the limitation on part of imposed on the amount of the clocking rate value,

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random access memory size and existense of intermediate expressions swelling phenomenon, an impossible practical task even for the state-of-the-art CAS like Axiom, Macsyma, Maple, Mathematica running on a high-performance computer to get an MDI directly, though each intermediate integral is still expressible in a closed form.

In [B95] it was suggested to use a computer algebra system to gain an insight of the exact structure of the solution of an $n-$dimensional integral using an algorithm-like heuristic, and then to produce a rigorous proof of the result (by using a CAS and/or 'paper-and-pensil' technique). For example, we have evaluated *exactly*, by guessing and proving, some n-dimensional integrals taken over an n-cube with arbitrary n, which are absent, amongst other handbooks, in the most extensive Brychkov–Prudnikov–Marichev tables.

Instead of direct numerical computation of an MDI using the QMC/DP techniques, which might be beyond straightforward computation, it is possible to evaluate a proper n-dimensional integral using a CAS and a *low*-performance computer, then subsitite the concrete values of the parametres, and then approximate the resulting (usually, compact) output using arbitrary presisions numerical procedures. For example, the following integral can be computed with ease:

$$\underbrace{\int_0^1 \cdots\cdots \int_0^1}_{1000 \text{ times}} \frac{dx_1 dx_2 ... dx_{1000}}{(1+x_1+x_2+...+x_{1000})} =$$

0.00199 66713 27518 13999 85774 90481 64370 96210 56060 56944
31841 58352 81371 44374 23342 99762 01349 45312 33924 94799
44702 35705 30035 58041 62346 30620 59447 48444 72653 78145
39069 83207 21333 79287 98038 42938 29153 09395 96275 92681
20582 62321 40765 90419 59954 55078 76059 12305 35585 33098...

The 248 above exact digits of the integral were taken in 58.1 seconds using a simplest Pentium Intel 286 based computer with 12 MHz clock rate, RAM of 640 Kb, and Derive computer algebra, version 2.59. The size of the abstract prevents us from giving the complete proof of the statement.

The integrands of computed by the author MDIs, up to now, involve the simplest multivariable functions only. There are, at least, three open questions : 1) What is the exact class of integrands involving the elementary functions, for which there exists an exact closed form expression (say, MDI is taken over n-cube) ? 2) Is it possible to find MDIs involving a special function(s), which still yield a closed form formulas (say, MDI is taken over n-cube) ? 3) What is the exact set of n-dimensional integration regions different from n-cube for which are still possible to find a closed form expression ?

Finally, it seems interesting to note that using of the explicit n-dimensional formulas for high-precision approximation of MDIs vs direct computation of MDIs even in 5-dimensional space can produce speed-up of 10,000, and the higher dimension of space to integrate the more dramatical speed-up one obtains.