Computer Algebra System as a Tool :

Exact evaluation of definite n–dimensional integrals with *undetermined* a priory integer n

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Extended Abstract

A *very* restricted number of n–dimensional integrals was taken up to now. So it is naturally to try to extend the list of such integrals as far as possible. Most of these integrals is due to Jacobi (over sphere), Dirichlet (over simplex and generalized ellipsoid[1]), Liuoville (over simplex and cube), Cauchy (over cube), N. Sonin (over sphere), E. Catalan (over spherical surface). The basic proof methods were induction, change of variables and differentiation wrt parameter.

At present an algorithm for evaluation of n–dimensional integrals with undetermined a priory n is still unknown.

Nevertheless, a computer algebra system (CAS) can be considered as an efficient enough tool for *exact* evaluation of n–dimensional definite integrals with *undetermined a priori* integer n over some 'regular enough' regions (RER) in Euclidean space $\mathbb{R}^n$. At present the author cannot give a strict definition of RER; by 'regular enough' region we understand a highly decomposable region R such that $R \subseteq \mathbb{R}^n$, $R = \cup_{i=1}^{k} R_i$, $R_i \subseteq \mathbb{R}^i$, $k \ll n$ at $n \gg 1$, and all the R_i are smooth. To support this statement, the author presents 2 of more than 15 integrals taken over n–cube and n–half–space [B96]. The main idea of the research is to recognize the exact answer for arbitrary integer n (using two heuristics) and then to construct the rigorous proof of it. Note that it might be possible to use a (specialized) proof development environment for getting the proof; it seems to be interesting that in this work the author used a *CAS itself* both *to get an idea* for a proper proof *and to construct* the proof.

Let C_n be direct product of n intervals [0,1]; H_n be direct product of n intervals $[0,\infty)$; $d\Omega_n = dx_1 dx_2 \ldots dx_n$, $\vec{a}\cdot\vec{x}$ is a dot product in $\mathbb{R}^n$ and a constant vector $\vec{a}$ has the components $a_j \neq 0$. Let also assume for the formula (1) the restrictions $\sum_{j=1}^{n} a_j > -\beta$, $\beta > -1$. Then (1)–(2) integrals over C_n and H_n regions hold (it means that the correct structures of r.h.s. were found and then rigorously proved, by proving corresponding identities for iterated derivatives) :

An example of integral over n–cube C_n :

$$\int_{C_n}\cdots\int ln(\beta+\vec{a}\cdot\vec{x})d\Omega_n = -\sum_{i=1}^{n}\frac{1}{i} + \frac{1}{n!\prod_{i=1}^{n}a_i}\sum_{\tau\in S_n}(-1)^{(n+\mathrm{rank}(\tau))}(\beta+\tau)^n ln(\beta+\tau)\,, \quad n \geq 1 \tag{1}$$

where $\Gamma(x)$ is Euler's gamma function; the second sum is taken over the set S_n which depends on dimension n of space $\mathbb{R}^n$ and components a_j of $\vec{a}$: $S_n = \left\{0,\ \sum_{j\in\Sigma} a_j \,\middle|\, \Sigma \in C\{1,2,\ldots,n\}\right\}$, and $C\{1,2,\ldots,n\}$ is the set of all the combinations of elements of the set $\{1,2,\ldots,n\}$. The domain of function rank(x) is exactly the elements of the above–defined set S_n, rank(0)=0 and rank(x) = number of items in x if $x \neq 0$.

An example of integral over n–half–space H_n $(b_j \neq 0;\ \ n,k,m \in \mathbb{N},\ \ k \geq n+m)$ **:**

$$\int_{H_n}\cdots\int \frac{(a_0+\vec{a}\cdot\vec{x})^m}{(b_0+\vec{b}\cdot\vec{x})^k}\ d\Omega_n = \frac{m!\ \ (k-n-m-1)!}{(k-1)!\ \ b_0^{(k-n-m)}\ \prod_{i=1}^{n}b_i}\ \sum_{j=0}^{m}\left(\binom{k-n-j-1}{m-j}\sum^{*}(j,n)\right), \tag{2}$$

[1] This region is defined as $\{\ (x_1/a_1)^{\beta_1} + (x_2/a_2)^{\beta_2} + \ldots (x_n/a_n)^{\beta_n} \leq 1 \ \ |\ \ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\ \}$.

where $\sum^*(m,n) = \sum_{k_1=1}^{n}(\frac{a_{k_1}}{b_{k_1}}) \ldots \sum_{k_{m-1}=k_{m-2}}^{n}(\frac{a_{k_{m-1}}}{b_{k_{m-1}}}) \sum_{k_m=k_{m-1}}^{n}(\frac{a_{k_m}}{b_{k_m}})$ and $\sum^*(0,n) = (\frac{a_0}{b_0})^n$.

Additional 'basic' DMIAMs over region C_n are taken exactly with the following integrands: $arctan(\beta+\vec{a}\cdot\vec{x})$, $arctanh(\beta+\vec{a}\cdot\vec{x})$, $arcsin(\vec{a}\cdot\vec{x})$, $arcsinh(\vec{a}\cdot\vec{x})$ and $erf(\beta+\vec{a}\cdot\vec{x})$ and some others; also over 10 integrals over H_n region are described in [B96].

At present an algorithm which is able to decide whether one integral with *arbitrary* number of variables can be converted into the other one is also unknown yet. The algorithms were described dealing with *fixed* number of variables 'only' [T90], [WZ92]. Nevertheless, as a rule an expert can recognize the case without failure almost always due to his intuition and some additional computations. Over 15 definite multiple integrals with arbitrary multiplicity (DMIAMs) which were *exactly* taken with aid of CAS are given in [B96]. None of these DMIAMs is present in the most extensive and new tables such as [PBM81] as well as in the set of others : [GR71], [D61], [H67c], [H64]. As an expert the author states these integrals cannot arise as a consequence of any transformations of the entries of the above–mentioned tables.

Moreover, it should be emphasized specially that the author was not be able to find the correct expressions for the above–mentioned integrals without the CRUCIAL aid of CAS *in view of an exorbitant amount of computations.* Hence, it seems some *really new* integrals are evaluated *due to crucial aid of CAS.*

The proposed technique for recognition of r.h.s of n–dimensional integral is based on joint using of two heuristics. In author's opinion, the first heuristic often brings a part of information faster than the second one.

<u>The first heuristic.</u> Compute the successive *indefinite* integrals in $\mathbb{R}^n$ with $n = 1, 2, 3, \ldots$ until a hypothetically right structure of summand including a summation–independent factor be guessed using steps I1—I4.

I1 Change *all* numbers the integrand under consideration into integration–independent parameters.

I2 Integrate the resulted parametrized indefinite integral.

I3 IF any polynomial in introduced parameter(s) (specifically, a non–zero constant) is detected in the corresponding definite integral THEN IF there exists only a pure polynomial part (PPP) in the output of the above–mentioned *in*definite integral THEN use PPP as in I4; ELSE throw off PPP: it will be cancelled after all the substitutions in the definite integral.

I4 Try to recognize in the output some known mathematical structures like a perfect power etc completely in spirit of steps D5, D6, D7, D8 of the second heuristic; express numeric constants via dimension of space (e.g. in $\mathbb{R}^3$ constant $\frac{1}{24}$ can be interpreted as $\frac{1}{(3+1)!} = \frac{1}{(DIM+1)!}$ etc).

<u>The second heuristic.</u> Compute the successive *definite* integrals in $\mathbb{R}^n$ with $n = 1, 2, 3, \ldots$ until a hypothetically right answer be guessed using steps D1—D8 :

D1 Change all numeric coefficients (including factor 1) and constants (if any) in the integrand under consideration into integration–independent parameters.

D2 Set $n = 1$. (It would be harder to recognize some known (sub)expressions at large n.)

D3 Evaluate with aid of some CAS the exact value of the DMIAM at given n.

D4 If there is a sum, expand each coefficient and each argument of functions involved in the expression of the computed sum term by term.

D5 Try to factorize each term of the sum (it also might lead to insight) in spirit of step D4.

D6 Try to restore correctly numbers and symbolic monoms which are reduced by CAS by default (e.g. one need to recognize $\frac{3}{4!}$ in $\frac{1}{8}$ or $\frac{a}{abc}$ in $\frac{1}{bc}$ and so on).

D7 Try to recognize some known math structure in the output terms like a binomial coefficient, a factorial, a perfect square/cube/n^{th}–power etc involving all the parameters or a part of them.[2] Try to rearrange the computed data towards some known things; grasp some pattern. Then after spending some time set $n = n + 1$ for checking or further guessing.

[2]There are tools inside some CASs for better guessing a mathematical structure. For example, in REDUCE one may use the operator STRUCTR — and the structure will be printed effectively as a tree. See also a remark on GFUN and GUESSS.

D8 If a reasonable confidence level in guessed answer is achieved then STOP and try to prove the formula, otherwise go to step D3.

It should be emphasized specially that the author could not guess the correct expressions for the above–mentioned integrals without the CRUCIAL aid of CAS *in view of an exorbitant amount of computations.* Actually integrator of Derive [RS95] was used very intensively at the steps D3, I2; factorizer at the steps D5, D6, D7, I4 and differentiation to search for and check iterated derivatives up while looking for a proof.

Remark. For recognition of the structure of constants of the above–mentioned more than 15 DMIAMs the author also tried to use jointly GUESSS procedure by Harm Derksen and GFUN package by Bruno Salvy & Paul Zimmermann (version of March 4, 1992) from MAPLE share library and had 4 successful cases and 15 failures for the given versions.

The main and obvious drawback of the described approach is necessity of computer–aided human recognition of some mathematical structures a part of which might be NOT known in advance e.g. such as multisum $\sum^*(m,n) = \sum_{k_1=1}^{n}(\frac{a_{k_1}}{b_{k_1}})\cdots\sum_{k_{m-1}=k_{m-2}}^{n}(\frac{a_{k_{m-1}}}{b_{k_{m-1}}})\ \sum_{k_m=k_{m-1}}^{n}(\frac{a_{k_m}}{b_{k_m}})$ in (2).

Up to now all the integrands of discovered by the author DMIAMs involve simple enough multivariable functions and all the integrals were taken over C_n and H_n regions only. So there are, at least, 3 *open questions:* **1)** What is the exact class of integrands involving elementary functions, for which still there exists an exact closed–form expression[3] (DMIAM is taken over C_n or H_n) ? **2)** What is the exact set of n–dimensional integration regions different from C_n and H_n for which it is still possible to find closed–form expressions ? **3)** Is it possible to get some *new* n–dimensional integrals involving more SPECIAL function(s) over some 'regular enough' regions ? The author has an impression this question could be answered *by combining* the technique described in this paper and that developed in [AM90] which enables us to integrate functions of hypergeometric type.

Of course, using these 'basic' DMIAMs it is possible by various transformations (specifically due to nonlinear substitutions) to generate *a number of quasi–new DMIAMs* which can be of value for different areas.

The author suggests to use the DMIAMs formulae *for the mathematical data base extension of computer algebra systems*, after checking them up by independent experts.

The explicit formulae for r.h.s. of DMIAMs (table lookup technique) would also lead to *dramatic* (often of order 10^2–10^4 even in $\mathbb{R}^3$) speed–up due to evaluation of symbolic sums instead of direct computations of the corresponding integrals [B95], [B95a], [B96] — because construction of an internal representation tree via the sums requires essentially less memory cells, though it is still of exponential complexity in number of nodes at the worst case [B96].

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